is obtained (called the *kinetic equation* corresponding to Eq. (A3.2.7)). If the last equation has an equilibrium, $f(a) = 0$ then $u(t, x, y, z) \equiv a$ is also a solution of the PDE (A3.2.7). If $u = a$ is an asymptotically stable equilibrium of the kinetic equation and B is convex then, under no flux boundary conditions, it is also an asymptotically stable solution of Eq. (A3.2.7), which means that solutions with initial values near to a stay near to a and tend to a as t tends to infinity. In this case, when the underlying space is 3D, no flux boundary conditions mean that the directional derivative of u in the direction of the normal vector of S is zero everywhere on S for $t \geq 0$. Eq. (A3.2.7) also may have nonconstant stationary solutions, that is, solutions $u(x, y, z)$ that do not depend on t. However, under the forementioned conditions no such solution can be asymptotically stable. For these results see Casten and Holland (1977,1978). If there is more than one substance diffusing and reacting with another then the last statement is no longer valid. We shall handle this problem in the next Section.

A3.3 Turing Bifurcation

Consider now two substances that react with each other (activating or inhibiting the production of each other) and diffuse in a spatial domain according to Fick's law. We assume a 2D bounded, connected spatial domain B with piecewise smooth boundary ∂B, and denote the respective densities at time t and point $(x, y) \in B$ by $u(t, x, y)$, $v(t, x, y)$ where x and y are Cartesian orthogonal coordinates. Then proceeding analogously to the single substance case in the previous Section, the dynamics is described by the system of reaction-diffusion equations

$$\frac{\partial u}{\partial t} = d_u \Delta u + f(u, v)$$
$$\frac{\partial v}{\partial t} = d_v \Delta v + g(u, v) , \qquad \text{(A3.3.1)}$$

where $d_u, d_v > 0$ are the respective diffusion coefficients, $f, g \in C^1$ are the reaction terms, and the Laplace delta is $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. No flux boundary conditions are attached; denoting the directional derivative in the direction orthogonal to the boundary ∂B by $\partial/\partial n$:

$$\frac{\partial u(t, x, y)}{\partial n} = \frac{\partial v(t, x, y)}{\partial n} = 0 , \ t \geq 0 , \ (x, y) \in \partial B . \qquad \text{(A3.3.2)}$$

The ODE system

$$\dot{u} = f(u, v) , \ \dot{v} = g(u, v) , \qquad \text{(A3.3.3)}$$

where the overdot denotes differentiation with respect to time t, is called the *kinetic system* attached to Eq. (A3.3.1). If $f(\bar{u}, \bar{v}) = g(\bar{u}, \bar{v}) = 0$ then the equilibrium $(\bar{u}, \bar{v})$ is also a constant solution of the PDE system (A3.3.1). However, contrary to the scalar case the asymptotic stability of $(\bar{u}, \bar{v})$ with respect to system (A3.3.3) does not necessarily imply its asymptotic stability with respect to the PDE system (A3.3.1).

DEFINITION A3.3.1. We say that a stationary solution $(\tilde{u}(x, y), \tilde{v}(x, y))$ of the problem (A3.3.1)-(A3.3.2) is *stable in the Lyapunov sense* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if for the initial value $(u(0, x, y), v(0, x, y))$ of a solution of the problem

$$\sup_B \left(|u(0, x, y) - \tilde{u}(x, y)| + |v(0, x, y) - \tilde{v}(x, y)|\right) < \delta$$

holds then for $t > 0$, $(x, y) \in B$ we have

$$|u(t, x, y) - \tilde{u}(x, y)| + |v(t, x, y) - \tilde{v}(x, y)| < \varepsilon ;$$

we say that this stationary solution is *asymptotically stable* if it is stable in the Lyapunov sense and there exists an $\eta > 0$ such that if

$$\sup_B \left(|u(0, x, y) - \tilde{u}(x, y)| + |v(0, x, y) - \tilde{v}(x, y)|\right) < \eta$$

then

$$\lim_{t \to \infty} \left(|u(t, x, y) - \tilde{u}(x, y)| + |v(t, x, y) - \tilde{v}(x, y)|\right) = 0 .$$

Even if B is convex for problem (A3.3.1)-(A3.3.2) the situation may arise that *a constant stationary solution is asymptotically stable with respect to the kinetic system (A3.3.3) but is unstable with respect to problem (A3.3.1)-(A3.3.2).* This possibility, discovered by Turing (1952) in his famous paper on morphogenesis is called *Turing* (or *diffusion driven) instability.* Spatially constant initial conditions are, at the same time, initial conditions with respect to the kinetic system, and the corresponding solution of the latter is obviously also a solution of the problem (A3.3.1)-(A3.3.2) depending only on the time t. Therefore, Turing instability means that solutions with constant initial values near to the constant solution tend to the latter as time tends to infinity while solutions corresponding to spatially nonconstant initial conditions arbitrarily near to the constant solution may tend away from it. This phenomenon is interesting because the general experience is that diffusion helps stability by evening out differences, and now the opposite happens, and it is also of interest because Turing instability may go together with the occurrence of a spatially nonconstant stationary solution, which is called a *pattern.*

First, necessary conditions will be deduced for the occurrence of Turing instability. Denote an equilibrium point of the kinetic system (A3.3.3) by $(\bar{u}, \bar{v})$, linearize the system at this point, denote the coefficient matrix of the linearized system by

$$A = \begin{bmatrix} f'_u(\bar{u}, \bar{v}) & f'_v(\bar{u}, \bar{v}) \\ g'_u(\bar{u}, \bar{v}) & g'_u(\bar{u}, \bar{v}) \end{bmatrix} ,$$

and assume that the eigenvalues of this matrix μ_1^0, μ_2^0 have negative real parts. This assumption implies the asymptotic stability of the equilibrium with respect to system (A3.3.3). The characteristic polynomial of A is

$$\mu^2 - (f'_u + g'_v)\mu + \det A \tag{A3.3.4}$$

so that from the remark following Theorem A1.1.2 our assumption is equivalent to saying that

$$f'_u + g'_v < 0 \ , \ \det A = f'_u g'_v - f'_v g'_u > 0 \tag{A3.3.5}$$

(here and in the sequel the arguments will not be written out; they are always $(\bar{u}, \bar{v})$). Now system (A3.3.1) is to be linearized at the constant stationary solution $(\bar{u}, \bar{v})$, the "general solution" of the linearized system with boundary conditions (A3.3.2) will be written out, and the stability of the $(0,0)$ solution of the linearized problem will be considered. By shifting the origin of the phase space into $(\bar{u}, \bar{v})$, introducing the coordinates $p = u - \bar{u}$, $q = v - \bar{v}$, writing out the equation in the new coordinates, and dropping the higher-order terms, we arrive at the linearized system

$$\begin{aligned} \frac{\partial p}{\partial t} &= d_u \Delta p + f'_u p + f'_v q \\ \frac{\partial q}{\partial t} &= d_v \Delta q + g'_u p + g'_v q \end{aligned} \tag{A3.3.6}$$

to which the boundary conditions

$$\frac{\partial p(t,x,y)}{\partial n} = \frac{\partial q(t,x,y)}{\partial n} = 0 \ , \ t \geq 0 \ , \ (x,y) \in \partial B \tag{A3.3.7}$$

are attached. The problem (A3.3.6)-(A3.3.7) is to be solved by Fourier's method. We suppose that the problem has a solution of the form $(T_1(t), T_2(t))R(x,y)$, substitute this into Eq. (A3.3.6), divide the first and the second equation by $T_1 R$ and $T_2 R$, respectively, and obtain

$$\begin{aligned} \frac{\dot{T}_1}{T_1} &= d_u \frac{\Delta R}{R} + f'_u + f'_v \frac{T_2}{T_1} \\ \frac{\dot{T}_2}{T_2} &= d_v \frac{\Delta R}{R} + g'_u \frac{T_1}{T_2} + g'_v \end{aligned}$$

or

$$\begin{aligned} \frac{1}{d_u}\left(\frac{\dot{T}_1}{T_1} - f'_u - f'_v \frac{T_2}{T_1}\right) &= \frac{\Delta R}{R} = -\lambda \\ \frac{1}{d_v}\left(\frac{\dot{T}_2}{T_2} - g'_u \frac{T_1}{T_2} - g'_v\right) &= \frac{\Delta R}{R} = -\lambda \ , \end{aligned} \tag{A3.3.8}$$

where λ is a constant because the first term depends only on t and the second only on x, y. First the boundary value problem

$$\Delta R = -\lambda R \ , \ \frac{\partial R(x,y)}{\partial n} = 0 \ , \ (x,y) \in \partial B \tag{A3.3.9}$$

is to be solved. It is known (see Vladimirov, 1967; or Evans, 1998) that (A3.3.9) has a countable set of nonnegative eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq ...$ with eigenfunctions $R_k(x,y)$, $(k = 0,1,2,...)$ such that the eigenfunctions belonging to different eigenvalues are orthogonal to each other. To be sure, the determination of the eigenvalues and the eigenfunctions might be a difficult problem depending on the geometry of B but for simple domains such as a rectangle, a rectangular triangle or a circle, it can be done explicitly. Second, the 2D ODE system for (T_1, T_2) is written out from Eq. (A3.3.8):

$$\begin{aligned} \dot{T}_1 &= (f'_u - \lambda d_u)T_1 + f'_v T_2 \\ \dot{T}_2 &= g'_u T_1 + (g'_v - \lambda d_v)T_2 \end{aligned}$$

or

$$\begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \end{bmatrix} = (A - \lambda D) \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} , \tag{A3.3.10}$$

where D =diag$[d_u, d_v]$ is the diffusion matrix. We have to substitute each eigenvalue of problem (A3.3.9) into Eq. (A3.3.10) for λ and solve the latter system. The characteristic polynomial of (A3.3.10) is

$$\det(A - \lambda D - \mu I) = \mu^2 - \mu(f'_u + g'_v - \lambda(d_u + d_v))$$

$$+ \det A - \lambda(d_u g'_v + d_v f'_u) + \lambda^2 d_u d_v \ . \tag{A3.3.11}$$

Using (A3.3.5) for $\lambda \geq 0$; Trace$(A - \lambda D) = f'_u + g'_v - \lambda(d_u + d_v)$ is negative and thus we may have instability only if

$$\det(A - \lambda D) = \det A - \lambda(d_u g'_v + d_v f'_u) + \lambda^2 d_u d_v \leq 0 \ . \tag{A3.3.12}$$

If the characteristic polynomial has a double root then as the product of the roots this determinant is positive, so that in this case stability prevails. Therefore, we may assume without loss of generality that for each λ_k $(k = 0,1,2,...)$ (A3.3.11) has two distinct roots μ_1^k, μ_2^k with eigenvectors s^{k1}, s^{k2}, respectively, and as a consequence, the general solution of (A.3.3.10) is

$$\begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix} = c_{k1} e^{\mu_1^k t} s^{k1} + c_{k2} e^{\mu_2^k t} s^{k2} \ ,$$

where c_{k1}, c_{k2} are arbitrary constants. This way, for each $k = 0,1,2,...$ we have obtained a solution of the boundary value problem (A3.3.6)-(A3.3.7):

$$R_k(x,y)(c_{k1} e^{\mu_1^k t} s^{k1} + c_{k2} e^{\mu_2^k t} s^{k2}) \ .$$

If the series

$$\sum_{k=0}^{\infty} R_k(x,y)(c_{k1}e^{\mu_1^k t}s^{k1} + c_{k2}e^{\mu_2^k t}s^{k2}) \tag{A3.3.13}$$

is convergent and can be substituted into Eq. (A3.3.6), then it is also a solution of this linear problem. If smooth initial conditions $(p(0,x,y), q(0,x,y)) = (P(x,y), Q(x,y))$ are given then, in order to determine the appropriate coefficients in the last series, the initial functions are to be expanded into generalized Fourier series:

$$\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} = \sum_{k=0}^{\infty} R_k(x,y)(c_{k1}s^{k1} + c_{k2}s^{k2}) .$$

With the coefficients determined this way we get the solution of the boundary-initial value problem:

$$\begin{bmatrix} p(t,x,y) \\ q(t,x,y) \end{bmatrix} = \sum_{k=0}^{\infty} R_k(x,y)(c_{k1}e^{\mu_1^k t}s^{k1} + c_{k2}e^{\mu_2^k t}s^{k2}) .$$

If for each $\lambda_k, (k = 0,1,2,...)$ system (A3.3.10) is asymptotically stable, then in the last series every term tends to zero exponentially as time tends to infinity and hence every solution of problem (A3.3.6)-(A3.3.7) tends to (0,0). From Casten and Holland (1977) the asymptotic stability of the constant solution $(\bar{u}, \bar{v})$ of the nonlinear problem (A3.3.1)-(A3.3.2) is implied. In order to have Turing instability system (A3.3.10) must be unstable for at least one λ_k. This means that (A3.3.12) must hold for some positive λs. The roots of the polynomial in (A3.3.12) are

$$\lambda^{1,2} = \frac{1}{2d_u d_v}\left(d_u g_v' + d_v f_u' \pm \sqrt{(d_u g_v' + d_v f_u')^2 - 4d_u d_v \det A}\right) .$$

If $d_u g_v' + d_v f_u' \le 0$, then no root is positive. We have positive roots only if

$$d_u g_v' + d_v f_u' > 0 \tag{A3.3.14}$$

and

$$(d_u g_v' + d_v f_u')^2 > 4d_u d_v \det A . \tag{A3.3.15}$$

In this case both roots are positive and distinct (a double positive root is of no use because then we do not have an interval where (A3.3.12) holds). This way we arrived at

THEOREM A3.3.1 *If the equilibrium solution $(\bar{u}, \bar{v})$ is Turing unstable then conditions (A3.3.5), (A3.3.14), and (A3.3.15) must hold.*

By the first condition of Eq. (A3.3.5) at least one of f_u', g_v' must be negative. By (A3.3.14) at least one of them must be positive. Hence, $f_u' g_v' < 0$. We may assume without loss of generality that

$$f_u' > 0,\ g_v' < 0 . \tag{A3.3.16}$$

This means that u acts as an activator (its increase is increasing its production) and v as an inhibitor (its increase is decreasing its production). Then by (A3.3.5) $0 < f_u' < -g_v'$ and by (A3.3.14) $f_u' > -g_v' d_u/d_v$. The last two inequalities imply that *Turing instability may occur only if $0 < d_u/d_v < 1$*, that is, *the diffusion coefficient of the inhibitor is larger than that of the activator*:

$$d_u < d_v . \tag{A3.3.17}$$

In the discussion that follows we assume that (A3.3.5) and (A3.3.14)-(A3.3.17) hold. Under these conditions $0 < \lambda^1 < \lambda^2$, so that if for some $k \ge 1$ the corresponding eigenvalue λ_k of Eq. (A3.3.9) falls into this interval, then (A3.3.12) holds, that is, $\mu_1^k \mu_2^k \le 0$, which implies that, for example, $\mu_1^k < 0 \le \mu_2^k$. As a consequence, the corresponding term in series (A3.3.13) does not tend to zero, that is, the constant solution is no longer asymptotically zero. If μ_2^k is positive then this term tends to infinity; if $\mu_2^k = 0$ then $R_k(x,y)s^{k2}$ is a spatially nonconstant stationary solution, that is, a *pattern*.

Suppose that one of the parameters in system (A3.3.1) is varied in such a way that this does not affect conditions (A3.3.5), that is, the equilibrium point $(\bar{u}, \bar{v})$ remains an asymptotically stable solution of the kinetic system (A3.3.3). This *bifurcation parameter* denoted in the sequel by b can be one of the diffusion coefficients, a measure of the spatial domain B (its diameter or area), or a parameter in the functions f and g. As the bifurcation parameter is varied the interval $[\lambda^1, \lambda^2] \subset \mathbb{R}_+$ sweeps through the positive axis λ, changing its length and in the process maybe engulfing some of the eigenvalues λ_k. *Suppose that for $b < b_0$ the interval $[\lambda^1, \lambda^2]$ does not contain any of the λ_k-s: $[\lambda^1, \lambda^2] \cap \{\lambda_1, \lambda_2, \lambda_3, ...\} = \emptyset$,the empty set, at $b = b_0$ we have $\lambda_i = \lambda^2$, say, and for $b > b_0$ the interval contains this eigenvalue: $\lambda_i \in (\lambda^1, \lambda^2)$; we say then that at b_0 the constant solution $(\bar{u}, \bar{v})$ undergoes a Turing bifurcation.* This means that for $b < b_0$ the constant solution is asymptotically stable with respect to problem (A3.3.1)-(A3.3.2), for $b > b_0$ it is unstable while it remains asymptotically stable with respect to the kinetic system, and at $b = b_0$ the linearized problem (A3.3.6)-(A3.3.7) has a spatially nonconstant stationary solution. Applying a theorem from Smoller (1983), under generic conditions one may prove that in this case *in a (possibly one-sided) neighborhood of b_0 the nonlinear problem (A3.3.1)-(A3.3.2) has a spatially nonconstant stationary solution, a pattern* (for a proof in case of a concrete model see Cavani and Farkas 1994).